Int. J. Solids Structures, 1970, Vol. 6, pp. 1035 to 1054. Pergamon Press. Printed in Great Britain

MEAN FIELD VARIATIONS IN A STATISTICAL SAMPLE OF HETEROGENEOUS LINEARLY ELASTIC SOLIDS†

M. J. BERAN and J. J. MCCOY‡

Towne School of Civil and Mechanical Engineering,
University of Pennsylvania, Philadelphia, Pennsylvania

Abstract—The problem discussed is the determination of the mean (i.e. statistical average) field quantities in a statistical sample of heterogeneous linearly elastic solids. A formulation is obtained that determines $\{u_i(\mathbf{x})\}$, where u_i denotes the displacement field and the braces indicate an ensemble average. This formulation is presented in terms of an infinite sequence of correlation functions associated with the usual material properties. While this formulation is exact only for the case involving infinite solids, an example is used to demonstrate that the effect of a boundary on the formulation is significant only within a layer of the boundary, the thickness of which is determined by the largest correlation length associated with the field describing the material properties.

The formulation is investigated in detail for the case of a statistically homogeneous and statistically isotropic sample of locally isotropic solids. An explicit form is obtained for the limiting case of weakly inhomogeneous solids. The case of the slowly varying field; i.e. variations in $\{u_i(\mathbf{x})\}$ which are slow relative to variations in the material properties; is also studied and it is shown that the use of effective material parameters result in a valid first order approximation for this case. The first order correction for this same limit is shown to be similar to that of the first strain gradient theory.

INTRODUCTION

THE problem to be discussed is the determination of the mean (i.e. statistical average) field quantities for an assemblage of linearly elastic solids, the individual members of which have material properties that may be described by statistically homogeneous random functions of position. This problem is of interest both because of its own intrinsic nature and because the results obtained may be applied to gain some insight into the validity of using theories which allow for a localized structure to study the response of a composite continuum.

Although a statistical interpretation to some problems involving a continuum, has met with a measure of success, there have been relatively few attempts to apply these ideas to problems involving an elastic continuum. In those cases in which a statistical view has been adopted, the assumption is usually made that all characteristic lengths associated with the spatial variations of the mean field quantities, L, are large compared to all characteristic lengths, l, associated with the spatial variations in the material properties. If this assumption is valid, the governing equations on the mean field are identical in form to the usual deterministic equations. The only difference is that the constant parameters that define the material properties are replaced by constant effective parameters. The problem of interest is then to predict these effective material parameters, or bounds on these effective parameters, from information on the random variations that occur in each member of the assemblage. In the case in which each member of the assemblage is an isotropic linearly

† This work was partially supported by the Army Research Office, Durham, N.C. DA-31-124-ARO-D-340, and by the National Science Foundation, Washington, D.C. GK530.

‡ Present address: Department of Civil Engineering and Mechanics, The Catholic University of America, Washington, D.C.

elastic solid, Beran and Molyneux [1] obtained bounds on the effective bulk modulus and McCoy [2] obtained bounds on the effective shear modulus. Kröner [3] obtained an explicit equation for the effective properties of an assemblage of anisotropic solids for a class of materials which he terms "perfectly disordered" solids. A much more extensive literature exists on determining effective properties, or bounds on effective properties, that is not based on any statistical ideas. Hashin [4] has presented a comprehensive review article on these solutions. In Beran [5] the linear elastic medium is discussed from a statistical point of view.

During the same period as the above work there has been a parallel independent study of the statistical problem in the Russian literature. Lomakin [6, 7] has written a series of papers applying perturbation techniques. Further work by Lomakin and other authors is presented in Lomakin [8]. The subject will be treated comprehensively in a book to be published by Lomakin [9].

For problems involved with actually determining the variation of mean field quantities, efforts have largely been restricted to the propagation of plane harmonic waves. Keller [10] and Karal and Keller [11] have studied the propagation of plane waves in an infinite weakly inhomogeneous medium by using perturbation theory. Their results show that in the limit of small perturbations the propagation is that predicted for the homogeneous medium with the propagation constant replaced by an effective propagation constant. The effective propagation constant is complex, however, and hence a spatially decaying wave is obtained. Knopoff and Hudson [12] and Hudson [13] have also studied the case of the propagation of plane harmonic waves.

In the present study the object is to obtain an equation governing the mean field quantities for a statistical sample of linearly elastic solids. We have previously studied this problem for dielectric materials (Beran and McCoy [14].) We begin with the displacement equations of motion and average these equations. This yields an equation for the mean displacement field which contains the term $\{C'_{ijkl}\varepsilon'_{kl}\}$, where the braces indicate ensemble averaging, a prime indicates the fluctuating part of the indicated quantity about its mean value, C_{ijkl} denotes the elastic moduli tensor, and ε_{kl} the linear strain tensor. Subtracting these averaged equations from the original equations gives equations for the fluctuating part of the displacement field. These latter equations are formally inverted to give ε'_{kl} in terms of the mean value of the displacement field and the results are finally substituted back into the averaged equation to give the desired equation for the mean displacement field.

In the next section the procedure outlined above is carried out. The single inversion that must be accomplished, which involves determining the response of a homogeneous linearly elastic solid with a known body force and homogeneous boundary conditions, is accomplished by introducing a Green's function.

In Section 2 detailed computations are carried out for the case in which each member of the assemblage is a locally isotropic solid of infinite extent. The results show that the governing equations on the mean field quantities are identical in form to those predicted for a homogeneous solid if one allows the possibility of non-local effects. It is shown how this equation leads to a sequence of approximate equations in the case in which all characteristic lengths associated with spatial variations in the mean field quantities are large compared to all characteristic lengths associated with spatial variations of the material properties.

In Section 3 a further restriction is introduced. We restrict our attention here to assemblages in which the statistical properties of the elastic moduli are homogeneous and isotropic. Here it is seen that all the information needed to characterize the assemblage, for purposes of calculating mean field quantities, is contained in six scalar functions. These functions are theoretically calculable in terms of correlation functions of all orders involving the material properties of members of the assemblage. Further, the entire set of equations is transformed from real space to Fourier transform space and a solution for the displacement field formally obtained in the transformed space. With the aid of this solution and two special cases, it is possible to express the limits of contracted transforms of these six functions at $k = 0$ and k approaching infinity, where k is the magnitude of the Fourier transform vector. Finally, perturbation limits for the case of weakly inhomogeneous solids are presented for these functions.

In Section 4 approximations to the problem studied in Section 3 are presented which are valid when $L/l \gg 1$. In particular the zeroth order approximation is contained in the familiar field equations for a homogeneous solid with material properties replaced by effective properties. For statistically homogeneous and isotropic assemblages these effective properties are contained in two easily measurable quantities. In the first order approximation additional terms involving fourth order derivatives of the displacement field arise. Here two additional constants are needed to characterize the assemblage. These additional constants have the units of length and are associated with the correlation lengths of the spatial fluctuations of the material properties. An identification is made between this first-order approximation and the linearized version of Toupin's strain gradient theory [15].

In the next section the question of equating ensemble averages with local volume averages is discussed. It is seen that such an equation is valid in the case in which $L/l \gg 1$. Thus, the conditions necessary for this equation are the same as those required for the approximations in Section 4 to be valid.

In the last section the case of a bounded solid is briefly discussed. Here it is argued that the effects of the boundary on the equations governing the mean field quantities are confined to a layer of the boundary. The thickness of the layer is dependent on l and it seems reasonable to assume that this thickness is of order l. Thus, in the case in which $L_B/l \gg 1$, where L_B is a characteristic length associated with the boundary, the equations derived on the basis of infinite bodies is a valid approximation throughout most of the solid.

1. EQUATION ON MEAN FIELD

The equations governing the response of a linearly elastic solid are given by

$$\partial_j(C_{ijkl}\varepsilon_{kl}) + F_i = 0 \tag{1}$$

where

$$\varepsilon_{kl} = u_{(k,l)} = \tfrac{1}{2}(\partial_l u_k + \partial_k u_l). \tag{2}$$

In the above the elastic moduli tensor, C_{ijkl}, is to be described by statistically homogeneous random functions of position. For convenience, the body force, F_i, is taken to vary with position in a non-random fashion. The displacement field is denoted by u_k and the strain field, or the symmetric part of the gradient of the displacement field, is denoted by ε_{kl}. Determination of the response of the solid requires the inversion of equation (1) subject to prescribed boundary conditions which we shall take to be described in a non-random sense.

It is desired to obtain an equation governing the mean value (i.e. ensemble average) of the displacement field. Ensemble averages are to be denoted by $\{\ \}$. Toward this end, we average equation (1) which gives

$$\{C_{ijkl}\}\partial_j\{\varepsilon_{kl}\}+\partial_j\{C'_{ijkl}\varepsilon'_{kl}\}+F_i = 0 \tag{3}$$

where a prime denotes the spatial fluctuations of the indicated quantity about its mean value. Next, equation (3) is subtracted from equation (1) which gives

$$\{C_{ijkl}\}\partial_j\varepsilon'_{kl}+(I-P)\partial_j(C'_{ijkl}\varepsilon'_{kl}) = -\partial_j(C'_{ijkl}\{\varepsilon_{kl}\}) \tag{4}$$

where I denotes the identity operator and P denotes the operation of taking an ensemble average. Equation (4) is viewed as the equation governing the fluctuating part of the displacement field; i.e. $\varepsilon'_{kl} = u'_{(k,l)}$, in terms of the right hand side which is taken to represent a known forcing term. It is to be inverted subject to boundary conditions which are homogeneous by virtue of the statement of the original problem. In order to carry out the solution of equation (4), use is made of an iteration procedure. Thus, we write

$$\varepsilon'_{kl} = \sum_{n=1} \varepsilon'^{(n)}_{kl} \tag{5}$$

where

$$\{C_{ijkl}\}\partial_j\varepsilon'^{(1)}_{kl} = -\partial_j(C'_{ijkl}\{\varepsilon_{kl}\}) \tag{6}$$

and

$$\{C_{ijkl}\}\partial_j\varepsilon'^{(n)}_{kl} = -(I-P)\partial_j(C'_{ijkl}\varepsilon'^{(n-1)}_{kl}), \qquad n > 1. \tag{7}$$

Each of the problems defined by equations (6) and (7) is equivalent to determining the response of a homogeneous linearly elastic solid to a prescribed forcing term. Again, the boundary conditions to be applied in each case are homogeneous.

Before discussing the solutions of these problems a comment on the validity of using the iteration procedure may be in order. It is clear that its use does limit the validity of the results to be obtained to some restricted class of problems. Unfortunately, it is not possible for us to give a precise definition of this class. We assume instead that if the spatial fluctuations in material properties are sufficiently well behaved, as is the prescribed forcing, F_i, then the iterated solution is valid. It should be emphasized with regard to this question that since we shall not truncate the series (5) in the main body of the report the validity of the results is not dependent on being able to assign a small perturbation parameter to the variations in the material properties.

The solution of the problems defined by equations (6) and (7), are formally carried out by introducing an appropriate Green's function. The appropriate Green's function will depend on the physical dimension of the body to be analyzed as well as the nature of the conditions to be satisfied on the boundary. It is clear, therefore, that the expression obtained for ε'_{kl} in terms of $\{\varepsilon_{kl}\}$, F_i and C_{ijkl} depends on what happens on the boundary of the problem being analyzed. Hence $\{C'_{ijkl}\varepsilon'_{kl}\}$ and, therefore, the equation obtained on $\{\varepsilon_{kl}\}$ by substitution in equation (3) will depend on the nature of the boundary of the body being analyzed. It should be emphasized that this dependence of the equation governing $\{\varepsilon_{kl}\}$ on the boundary of the problem to which it is to be applied is not a result of the method used to obtain the equation. Rather it is inherent in the nature of the problem. It is therefore inappropriate, perhaps, to denote such an equation, a field equation. We shall return to the question of the dependence of the equation governing $\{\varepsilon_{kl}\}$ on boundary conditions after the equations are derived.

The Green's function for elasticity problems which we designate by $K_{klm}(\mathbf{x}, \mathbf{x}_1)$ is defined by the equation

$$\{C_{ijkl}\}\partial_j K_{klm} + \delta(\mathbf{x}-\mathbf{x}_1)\delta_{im} = 0 \tag{8}$$

where $\delta(\mathbf{x}-\mathbf{x}_1)$ is a three-dimensional Dirac delta function and δ_{im} is the Kronecker delta. The boundary conditions to use in conjunction with equation (8) are the homogeneous conditions of the same type as those of the specific problem being studied. It might be noted that K_{klm} is symmetric with respect to the first two indices. Using this Green's function, the solutions of the equations (6) and (7) are

$$\varepsilon_{kl}^{\prime(1)}(\mathbf{x}) = \int_{\mathbf{x}_1} K_{klr}(\mathbf{x}, \mathbf{x}_1)\partial_s^{(1)}[C'_{rspq}(\mathbf{x}_1)\{\varepsilon_{pq}(\mathbf{x}_1)\}]\,\mathrm{d}\mathbf{x}_1 \tag{9}$$

and

$$\varepsilon_{kl}^{\prime(n)}(\mathbf{x}) = (I-P)\int_{\mathbf{x}_1} K_{klr}(\mathbf{x}, \mathbf{x}_1)\partial_s^{(1)}[C'_{rspq}(\mathbf{x}_1)\varepsilon_{pq}^{\prime(n-1)}(\mathbf{x}_1)]\,\mathrm{d}\mathbf{x}_1, \qquad n > 1 \tag{10}$$

In these expressions $\partial_s^{(1)}$ denotes differentiation in $\mathbf{x}_1$ space.

Next, we calculate $\{C'_{ijkl}\varepsilon'_{kl}\}$. From equations (5), (9), (10)

$$\{C'_{ijkl}\varepsilon'_{kl}\} = \sum_n \{C'_{ijkl}\varepsilon_{kl}^{\prime(n)}\}, \tag{11}$$

$$\{C'_{ijkl}\varepsilon_{kl}^{\prime(1)}\} = \int_{\mathbf{x}_1} K_{klr}(\mathbf{x}, \mathbf{x}_1)\partial_s^{(1)}[\{C'_{ijkl}(\mathbf{x})\ C'_{rspq}(\mathbf{x}_1)\}\{\varepsilon_{pq}(\mathbf{x}_1)\}]\,\mathrm{d}\mathbf{x}_1$$

and

$$\{C'_{ijkl}\varepsilon_{kl}^{\prime(n)}\} = \int_{\mathbf{x}_1} K_{klr}(\mathbf{x}, \mathbf{x}_1)\partial_s^{(1)}[\{C'_{ijkl}(\mathbf{x})C'_{rspq}(\mathbf{x}_1)\varepsilon_{pp}^{\prime(n-1)}(\mathbf{x}_1)\}]\,\mathrm{d}\mathbf{x}_1 \qquad n > 1. \tag{12}$$

Successive substitution of equations (9) and (10) into equation (12) allows the expression of $\{C'_{ijkl}\varepsilon_{kl}^{\prime(n)}\}$ in terms of an integro-differential operator acting on $\{\varepsilon_{pq}\}$. The form of the higher order terms is obtained from an obvious extrapolation of the forms for $n = 2$ and $n = 3$.

$$\{C'_{i_1i_2i_3i_4}\varepsilon_{i_3i_4}^{(2)}\} = \int_{\mathbf{x}_2}\int_{\mathbf{x}_2} K_{i_3i_4i_5}(\mathbf{x}, \mathbf{x}_1)\partial_{i_6}^{(1)}[K_{i_7i_8i_9}(\mathbf{x}, \mathbf{x}_2)\partial_{i_{10}}^{(2)}[\{C'_{i_1i_2i_3i_4}(\mathbf{x})C'_{i_5i_6i_7i_8}(\mathbf{x}_1)C'_{i_9i_{10}i_{11}i_{12}}(\mathbf{x}_2)\}$$
$$\times\ \{\varepsilon_{i_{11}i_{12}}(\mathbf{x}_2)\}]]\,\mathrm{d}\mathbf{x}_2\,\mathrm{d}\mathbf{x}_1,$$

$$\{C'_{i_1i_2i_3i_4}\varepsilon_{i_3i_4}^{\prime(3)}\} = \int_{\mathbf{x}_1}\int_{\mathbf{x}_2}\int_{\mathbf{x}_3} K_{i_3i_4i_5}(\mathbf{x}, \mathbf{x}_1)\partial_{i_6}^{(1)}[K_{i_7i_8i_9}(\mathbf{x}_1, \mathbf{x}_2)\partial_{i_{10}}^{(2)}[K_{i_{11}i_{12}i_3}(\mathbf{x}_2, \mathbf{x}_3)\partial_{i_{14}}^{(3)}[$$
$$\{C'_{i_1i_2i_3i_4}(\mathbf{x})C'_{i_5i_6i_7i_8}(\mathbf{x}_1)C'_{i_9i_{10}i_{11}i_{12}}(\mathbf{x}_2)C'_{i_{13}i_{14}i_{15}i_{16}}(\mathbf{x}_3)\} - \{C_{i_1i_2i_3i_4}(\mathbf{x})C'_{i_5i_6i_7i_8}(\mathbf{x}_1)\}$$
$$\times\{C'_{i_9i_{10}i_{11}i_{12}}(\mathbf{x}_2)C'_{i_{13}i_{14}i_{15}i_{16}}(\mathbf{x}_3)\}]\{\varepsilon_{i_{15}i_{16}}\}]]]\,\mathrm{d}\mathbf{x}_3\,\mathrm{d}\mathbf{x}_2\,\mathrm{d}\mathbf{x}_1. \tag{13}$$

The statistics of the ensemble enter these operators according to the following combinations of correlation functions.

$$
\begin{aligned}
\Gamma^{(1)}_{i_1\ldots i_8}(\mathbf{x}_1,\mathbf{x}_2) &= \{C'_{i_1\ldots i_4}(\mathbf{x}_1)C'_{i_5\ldots i_8}(\mathbf{x}_2)\}\\
\Gamma^{(2)}_{i_1\ldots i_{12}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= \{C'_{i_1\ldots i_4}(\mathbf{x}_1)C'_{i_5\ldots i_8}(\mathbf{x}_2)C'_{i_9\ldots i_{12}}(\mathbf{x}_3)\}\\
\Gamma^{(3)}_{i_1\ldots i_{16}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) &= \{C'_{i_1\ldots i_4}(\mathbf{x}_1)C'_{i_5\ldots i_8}(\mathbf{x}_2)C'_{i_9\ldots i_{12}}(\mathbf{x}_3)C'_{i_{13}\ldots i_{16}}(\mathbf{x}_4)\}\\
&\quad -\{C'_{i_1\ldots i_4}(\mathbf{x}_1)C'_{i_5\ldots i_8}(\mathbf{x}_2)\}\{C'_{i_9\ldots i_{12}}(\mathbf{x}_3)C'_{i_{13}\ldots i_{16}}(\mathbf{x}_4)\}\\
\Gamma^{(4)}_{i_1\ldots i_{20}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\mathbf{x}_5) &= \{C'_{i_1\ldots i_4}(\mathbf{x}_1)C'_{i_5\ldots i_8}(\mathbf{x}_2)C'_{i_9\ldots i_{12}}(\mathbf{x}_3)C'_{i_{13}\ldots i_{16}}(\mathbf{x}_4)C'_{i_{17}\ldots i_{20}}(\mathbf{x}_5)\}\\
&\quad -\{C'_{i_1\ldots i_4}(\mathbf{x}_1)C'_{i_5\ldots i_8}(\mathbf{x}_2)C_{i_9\ldots i_{12}}(\mathbf{x}_3)\}\{C'_{i_{13}\ldots i_{16}}(\mathbf{x}_4)C'_{i_{17}\ldots i_{20}}(\mathbf{x}_5)\}\\
&\quad -\{C'_{i_1\ldots i_4}(\mathbf{x}_1)C'_{i_5\ldots i_8}(\mathbf{x}_2)\}\{C'_{i_9\ldots i_{12}}(\mathbf{x}_3)C'_{i_{13}\ldots i_{16}}(\mathbf{x}_4)C'_{i_{17}\ldots i_{20}}(\mathbf{x}_5)\}.
\end{aligned}
\tag{14}
$$

Again, the extrapolation for the higher order combinations is clear.

Combining terms of all orders allows us to finally write

$$\{C'_{ijkl}\varepsilon'_{kl}\} = A_{ijkl}(\mathbf{x},\mathbf{x}_1)\{\varepsilon_{kl}(\mathbf{x}_1)\} \tag{15}$$

where $A_{ijkl}(\mathbf{x},\mathbf{x}_1)$ represents an infinite sum of integro-differential operators which contain correlation functions of all orders involving combinations of the material properties. Substituting equation (15) into equation (3) gives the final equation on $\{\varepsilon_{kl}\}$. We have

$$\{C_{ijkl}(\mathbf{x})\}\partial_j\{\varepsilon_{kl}(\mathbf{x})\}+\partial_j[A_{ijkl}(\mathbf{x},\mathbf{x}_1)\{\varepsilon_{kl}(\mathbf{x}_1)\}]+F_i(\mathbf{x}) = 0. \tag{16}$$

2. ASSEMBLAGE OF LOCALLY ISOTROPIC SOLIDS OF INFINITE EXTENT

In this section we shall specialize equation (16) to the case in which every body in the assemblage defining the statistical sample is a locally isotropic solid of infinite extent.† Thus the elastic moduli tensor $C_{ijkl}(\mathbf{x})$ is described by the general fourth-order isotropic tensor which is symmetric with respect to interchange of first two indices and we write

$$C_{ijkl}(\mathbf{x}) = \lambda(\mathbf{x})\delta_{ij}\delta_{kl}+\mu(\mathbf{x})(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}). \tag{17}$$

In this expression λ and μ are the familiar Lamé parameters which in the present problem are taken to be described by random functions of position which have constant mean values.

For the isotropic solid, the Green's function introduced in the last section is readily obtained by introducing a potential function. See for example, Love [16], p. 185. In the case of an infinite solid the desired Green's function is a solution of Kelvin's problem (see Love [16]) and we may immediately write it as

$$K_{ijk}(\mathbf{x},\mathbf{x}_1) = \frac{1}{8\pi\{\lambda+2\mu\}}\left[[\partial_i(r^{-1})\delta_{jk}+\partial_j(r^{-1})\delta_{ik}]-\frac{\{\lambda+\mu\}}{\{\mu\}}[\partial_i\partial_j(r^{-1})]r_k\right] \tag{18}$$

where

$$r_i = x_i - x_{1i}, \qquad r = |r_i|, \quad \text{and} \quad \partial_j = \partial/\partial x_j = \partial/\partial r_j.$$

† Strictly speaking some convergence difficulties can be expected for the case of an infinite solid unless one takes the fluctuations in the elastic moduli tensor to be zero outside of some finite volume. Once the problem has been solved, it is then permissible to allow this volume to become infinite as is required in the case of homogeneous statistics.

We should now like to substitute this expression for $K_{ijk}(\mathbf{x}, \mathbf{x}_1)$ into the equations defining the integro-differential operators $A_{ijkl}(\mathbf{x}, \mathbf{x}_1)$. First we notice that wherever $K_{ijk}(\mathbf{x}, \mathbf{x}_1)$ appears, it appears in the combination

$$\int K_{ijk}(\mathbf{x}, \mathbf{x}_1)\partial_l^{(1)}[\mathscr{A}(\mathbf{x}_1)]\,\mathrm{d}\mathbf{x}_1 \tag{19}$$

where $\mathscr{A}(\mathbf{x}_1)$ denotes a general tensor function of $\mathbf{x}_1$. We should like to express this integro-differential operation on $\mathscr{A}(\mathbf{x}_1)$ by an integral operation and a boundary term which we might evaluate by invoking Green's theorem. Thus, we write the above integral as

$$\int \partial^{(1)}[K_{ijk}(\mathbf{x}, \mathbf{x}_1)\mathscr{A}(\mathbf{x}_1)]\,\mathrm{d}\mathbf{x}_1 - \int [\partial_l^{(1)}K_{ijk}(\mathbf{x}, \mathbf{x}_1)]\mathscr{A}(\mathbf{x}_1)\,\mathrm{d}\mathbf{x}_1. \tag{20}$$

There is a danger in doing this in that, although the integral as it appears in equation (19) may exist, the integrals appearing in equation (20) need not exist when taken separately. Expanding $\partial_l^{(1)}K_{ijk}(\mathbf{x}, \mathbf{x}_1)$ gives

$$\begin{aligned}\partial_l^{(1)}K_{ijk}(\mathbf{x}, \mathbf{x}_1) &= -\partial_l K_{ijk}(\mathbf{x}, \mathbf{x}_1)\\ &= -\frac{1}{8\pi\{\lambda+2\mu\}}\Bigg[[\partial_l\partial_i(r^{-1})\delta_{jk} + \partial_l\partial_j(r^{-1})\delta_{ik}]\\ &\quad - \frac{\{\lambda+\mu\}}{\{\mu\}}[[\partial_i\partial_j\partial_k(r^{-1})]r_k + \partial_i\partial_j(r^{-1})\delta_{kl}]\Bigg].\end{aligned} \tag{21}$$

From this expression it may immediately be seen that each term in $\partial_l^{(1)}K_{ijk}(\mathbf{x}, \mathbf{x}_1)$ becomes singular as (r^{-3}) as r approaches zero. Thus the second integral in equation (20) cannot exist in the usual sense unless $\mathscr{A}(\mathbf{x}_1)$ goes to zero as $\mathbf{x}_1$ approaches $\mathbf{x}$. In the present problem this is not the case, but $\mathscr{A}(\mathbf{x}_1)$ does approach a value at $\mathbf{x}_1 = \mathbf{x}$ in a manner that is independent of direction. With this information it is possible to show that the volume integral in equation (20) does exist in a "Cauchy principal value" sense. This is, if one introduces spherical coordinates centered at $\mathbf{x}_1 = \mathbf{x}$ and integrates first with respect to angular coordinates, one finds that the strong singularity at $\mathbf{x}_1 = \mathbf{x}$ vanishes.

The first integral we evaluate by invoking the divergence theorem. Thus, it is written

$$\int_s n_l K_{ijk}(\mathbf{x}, \mathbf{x}_1)\mathscr{A}(\mathbf{x}_1)\,\mathrm{d}\mathbf{x}_{1_s}$$

where n_l is an outward unit normal and the integration is over two spheres concentric about the point $\mathbf{x}_1 = \mathbf{x}$ at which $K_{ijk}(\mathbf{x}, \mathbf{x}_1)$ becomes singular. One sphere is taken to be unboundedly large and the second vanishingly small. The integral over the large sphere vanishes provided $\mathscr{A}(\mathbf{x})$ falls off rapidly enough as the radius of the sphere becomes large. This indeed is the case in the present study. The integral over the small sphere may be directly calculated by introducing spherical coordinates. The result is

$$-\frac{\mathscr{A}(\mathbf{x})}{30\{\mu\}\{\lambda+2\mu\}}[\{3\lambda+8\mu\}(\delta_{il}\delta_{jk}+\delta_{lj}\delta_{ik}) - 2\{\lambda+\mu\}\delta_{lk}\delta_{ij}]$$

and, therefore, equation (20) may be written as

$$\mathscr{A}(\mathbf{x})M_{ijkl} + \int_{\mathbf{x}_1} N_{ijkl}(\mathbf{x}, \mathbf{x}_1)\mathscr{A}(\mathbf{x}_1)\,\mathrm{d}\mathbf{x}_1 \tag{22}$$

where

$$M_{ijkl} = \frac{-1}{30\{\mu\}\{\lambda+2\mu\}}[\{3\lambda+8\mu\}(\delta_{il}\delta_{jk}+\delta_{lj}\delta_{ik})-2\{\lambda+\mu\}\delta_{lk}\delta_{il}] \tag{23}$$

and

$$N_{ijkl} = \partial_l K_{ijk}(\mathbf{x}, \mathbf{x}_1) = -\partial_l^{(1)} K_{ijk}(\mathbf{x}, \mathbf{x}_1). \tag{24}$$

With these expressions we may write

$$A_{ijkl}(\mathbf{x}, \mathbf{x}_1) = \sum_{n=1}^{\infty} A_{ijkl}^{(n)}(\mathbf{x}, \mathbf{x}_1) \tag{25}$$

where

$$\begin{aligned} A_{ijkl}^{(1)}\{\varepsilon_{kl}\} &= \int_{\mathbf{x}_1} K_{pqr}(\mathbf{x}, \mathbf{x}_1)\partial_s^{(1)}[\{C'_{ijpq}(\mathbf{x})C'_{rskl}(\mathbf{x}_1)\}\{\varepsilon_{kl}(\mathbf{x}_1)\}]\,\mathrm{d}\mathbf{x}_1 \\ &= M_{pqrs}\{C'_{ijpq}(\mathbf{x})C'_{rskl}(\mathbf{x})\}\{\varepsilon_{kl}(\mathbf{x})\} + \int_{\mathbf{x}_1} N_{pqrs}(\mathbf{x}, \mathbf{x}_1)\{C'_{ijpq}(\mathbf{x})C'_{rskl}(\mathbf{x})\}\{\varepsilon_{kl}(\mathbf{x}_1)\}\,\mathrm{d}\mathbf{x}_1 \\ A_{ijkl}^{(2)}\{\varepsilon_{kl}\} &= \int_{\mathbf{x}_1}\int_{\mathbf{x}_2} K_{pqr}(\mathbf{x}, \mathbf{x}_1)\partial_s^{(1)}[K_{tuv}(\mathbf{x}_1, \mathbf{x}_2)\partial_w^{(2)}[\{C'_{ijpq}(\mathbf{x})C'_{rstu}(\mathbf{x}_1)C'_{vwkl}(\mathbf{x}_2)\} \\ &\quad \times \{\varepsilon_{kl}(\mathbf{x}_2)\}]]\,\mathrm{d}\mathbf{x}_2\,\mathrm{d}\mathbf{x}_1 \\ &= M_{tuvw}M_{pqrs}\{C'_{ijpq}(\mathbf{x})C'_{rstu}(\mathbf{x})C'_{vwkl}(\mathbf{x})\}\{\varepsilon_{kl}(\mathbf{x})\} \\ &\quad + \int_{\mathbf{x}_1}\Big[M_{tuvw}N_{pqrs}(\mathbf{x}, \mathbf{x}_1)\{C'_{ijpq}(\mathbf{x})C'_{rstu}(\mathbf{x}_1)C'_{vwkl}(\mathbf{x}_1)\} \\ &\quad + \int_{\mathbf{x}_2} K_{pqr}(\mathbf{x}, \mathbf{x}_2)\partial_s^{(2)}[N_{tuvw}(\mathbf{x}_2, \mathbf{x}_1)\{C'_{ijpq}(\mathbf{x})C'_{rstu}(\mathbf{x}_2)C'_{vwkl}(\mathbf{x}_1)\}]\Big]\{\varepsilon_{kl}(\mathbf{x}_1)\}\,\mathrm{d}\mathbf{x}_1. \end{aligned} \tag{26}$$

Similar manipulations will now give the remaining terms in the series defining the integral differential operator $A_{ijkl}(\mathbf{x}, \mathbf{x}_1)$. The form of these terms is apparent from the above. Notice that upon combining we may finally write

$$A_{ijkl}\{\varepsilon_{kl}\} = D_{ijkl}(\mathbf{x})\{\varepsilon_{kl}(\mathbf{x})\} + \int_{\mathbf{x}_1} E_{ijkl}(\mathbf{x}, \mathbf{x}_1)\{\varepsilon_{kl}(\mathbf{x}_1)\}\,\mathrm{d}\mathbf{x}_1 \tag{27}$$

where D_{ijkl} and E_{ijkl} are algebraic quantities. From equations (26) and (27).

$$D_{ijkl}(\mathbf{x}) = M_{pqrs}\{C'_{ijpq}(\mathbf{x})C'_{rskl}(\mathbf{x})\} + M_{tuvw}M_{pqrs}\{C'_{ijpq}(\mathbf{x})C'_{rstu}(\mathbf{x})C'_{vwkl}(\mathbf{x})\} + \ldots \tag{28}$$

and

$$\begin{aligned} E_{ijkl}(\mathbf{x}, \mathbf{x}_1) &= N_{pqrs}(\mathbf{x}, \mathbf{x}_1)\{C'_{ijpq}(\mathbf{x})C'_{rskl}(\mathbf{x}_1)\} + M_{tuvw}N_{pqrs}(\mathbf{x}, \mathbf{x}_1)\{C'_{ijpq}(\mathbf{x})C'_{rstu}(\mathbf{x}_1)C'_{vwkl}(\mathbf{x}_1)\} \\ &\quad + \int_{\mathbf{x}_2} K_{pqr}(\mathbf{x}, \mathbf{x}_2)\partial_s^{(2)}[N_{tuvw}(\mathbf{x}_2, \mathbf{x}_1)\{C'_{ijpq}(\mathbf{x})C'_{rstu}(\mathbf{x}_2)C'_{vwkl}(\mathbf{x}_1)\}]\,\mathrm{d}\mathbf{x}_2 + \ldots \end{aligned} \tag{29}$$

Finally a great deal of simplification can be introduced in the expressions for D_{ijkl} and E_{ijkl} by expressing C_{ijkl} as indicated in equation (17). We find for example

$$\begin{aligned} -\{\lambda+2\mu\}D_{ijkl}(\mathbf{x}) &= \left(\{\lambda'^2\} + \frac{4}{3}\{\lambda'\mu'\} - \frac{4\{\lambda+\mu\}}{15\{\mu\}}\{\mu'^2\}\right)\delta_{ij}\delta_{kl} \\ &\quad + \frac{2\{3\lambda+8\mu\}}{15\{\mu\}}\{\mu'^2\}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \ldots \end{aligned} \tag{30}$$

and

$$2\pi\{\lambda+2\mu\}E_{ijkl}(\mathbf{x},\mathbf{x}_1) = -C_{\lambda\mu}(\mathbf{x},\mathbf{x}_1)\frac{\delta_{ij}\alpha_{kl}}{r^3} - C_{\mu\lambda}(\mathbf{x},\mathbf{x}_1)\frac{\alpha_{ij}\delta_{kl}}{r^3}$$
$$-C_{\mu\mu}(\mathbf{x},\mathbf{x}_1)\left[\frac{1}{2}\left(\frac{\delta_{jk}\alpha_{il}}{r^3}+\frac{\delta_{ik}\alpha_{jl}}{r^3}+\frac{\delta_{jl}\alpha_{ik}}{r^3}+\frac{\delta_{il}\alpha_{jk}}{r^3}\right)\right.$$
$$\left.-\frac{\{\lambda+\mu\}}{\{\mu\}}\left(\frac{\alpha_{ij}\delta_{kl}}{r^3}+\frac{3}{2}\frac{\beta_{ijkl}}{r^3}\right)\right]+\dots \quad (31)$$

In the above,

$$C_{\lambda\mu}(\mathbf{x},\mathbf{x}_1) = \{\lambda'(\mathbf{x})\mu'(\mathbf{x}_1)\},$$
$$C_{\mu\lambda}(\mathbf{x},\mathbf{x}_1) = \{\mu'(\mathbf{x})\lambda'(\mathbf{x}_1)\},$$
$$C_{\mu\mu}(\mathbf{x},\mathbf{x}_1) = \{\mu'(\mathbf{x})\mu'(\mathbf{x}_1)\},$$
$$\alpha_{ij} = \delta_{ij}-3r_ir_j/r^2,$$

and

$$\beta_{ijkl} = 10r_ir_jr_kr_l/r^4-2\delta_{ij}r_kr_l/r^2-(\delta_{il}r_jr_k/r^2+\delta_{jl}r_ir_k/r^2+\delta_{ik}r_jr_l/r^2+\delta_{jk}r_ir_l/r^2).$$

We now write equation (16) in the form

$$\partial_j\left[\int_{\mathbf{x}_1}\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)\{\varepsilon_{kl}(\mathbf{x}_1)\}\,d\mathbf{x}_1\right]+F_i = 0, \quad (32)$$

where

$$\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1) = [\{C_{ijkl}\}+D_{ijkl}(\mathbf{x}_1)]\delta(\mathbf{x}-\mathbf{x}_1)+E_{ijkl}(\mathbf{x},\mathbf{x}_1). \quad (33)$$

Recognizing that the ensemble average of the stress field, $\{\tau_{ij}\}$, is given by the bracketed quantity in equation (32), we may express the equations governing the mean stress and mean strain fields in an assemblage of linearly elastic solids in the following form

$$\partial_i\{\tau_{ij}\}+F_i = 0 \quad (34a)$$

$$\{\tau_{ij}(\mathbf{x})\} = \int_{\mathbf{x}_1}\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)\{\varepsilon_{kl}(\mathbf{x}_1)\}\,d\mathbf{x}_1 \quad (34b)$$

$$\{\varepsilon_{kl}\} = \tfrac{1}{2}(\partial_k\{u_l\}+\partial_l\{u_k\}). \quad (34c)$$

Thus we see, that the formulation governing the mean fields in an assemblage of linearly elastic solids is the same as that governing the field in a single linearly elastic solid if one allows for the presence of non-local effects. That is, the mean stress at a point is related to the mean strain at every point. It might be pointed out that although every body in the statistical sample might be locally isotropic, this need not result in an isotropic tensor for $\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)$. An additional restriction, i.e. that the statistics are also isotropic, is needed for us to conclude that $\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)$ is an isotropic tensor. The consequences of this assumption will be studied in the next section.

First, however, we should like to consider the formulation given by equation (34). Investigation of $\mathscr{C}_{ijkl}$ shows that in every term a correlation function appears, (or a function of correlation functions) appears, involving the $\mathscr{C}_{ijkl}$ field. The points located by $\mathbf{x}$ and $\mathbf{x}_1$ appear in these correlation functions, (or functions of correlation functions). This allows us to assume, since there is some characteristic length, l_c, associated with the fluctuations in C_{ijkl}, that $\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)$ is negligible for $|\mathbf{x}-\mathbf{x}_1| > l_c$. Thus, we conclude that the main contribution to the integral over $\mathbf{x}_1$ space appearing in equation (34b) comes from a region in a vicinity of the point $\mathbf{x}$ of characteristic dimension l_c. Consider now expanding $\{\varepsilon_{kl}(\mathbf{x}_1)\}$ in a power series about the point $\mathbf{x}$. We write

$$\{\varepsilon_{kl}(\mathbf{x}_1)\} = \{\varepsilon_{kl}(\mathbf{x})\}+(x_{1m}-x_m)\partial_m\{\varepsilon_{kl}(\mathbf{x})\}+\frac{(x_{1m}-x_m)(x_{1n}-x_n)}{2}\partial_m\partial_n\{\varepsilon_{kl}(\mathbf{x})\}. \quad (35)$$

Substituting equation (35) into equation (34b) yields

$$\{\tau_{ij}(\mathbf{x})\} = \left[\int_{\mathbf{x}_1}\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)\,\mathrm{d}\mathbf{x}_1\right]\{\varepsilon_{kl}(\mathbf{x})\}+\left[\int_{\mathbf{x}_1}\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)(x_{im}-x_m)\,\mathrm{d}\mathbf{x}_1\right]\partial_m\{\varepsilon_{kl}(\mathbf{x})\}+\ldots \quad (36)$$

or

$$\{\tau_{ij}(\mathbf{x})\} = C^*_{ijkl}\{\varepsilon_{kl}(\mathbf{x})\}+\mathscr{D}_{ijklm}[\partial_m\{\varepsilon_{kl}(\mathbf{x})\}]+\mathscr{E}_{ijklmn}[\partial_m\partial_n\{\varepsilon_{kl}(\mathbf{x})\}]+\ldots. \quad (37)$$

For problems in which a change in $\{\varepsilon_{kl}(\mathbf{x})\}$ over the characteristic length, l_c, is small it is possible to truncate equation (37). Truncation after a single term formally equates the problem of finding the mean field in an assemblage of solids to that of finding the field in a single solid. For this reason it appears reasonable to denote C^*_{ijkl} as the "effective" elastic moduli tensor. It should be noted that C^*_{ijkl} is easily measured. In fact it is the quantity that is determined when one obtains the elastic moduli tensor of any material that has local variations in material properties. Keeping more and more terms results in obtaining higher and higher order approximations. The higher order approximation of greatest interest will be the first correction. We shall look further at this correction after we have assumed homogeneous and isotropic statistics. This, we shall see, requires retention of the third order term in equation (37).

3. STATISTICALLY HOMOGENEOUS AND ISOTROPIC ASSEMBLAGE OF SOLIDS

In this section, we shall further restrict the problem being studied to one that involves a statistically homogeneous and isotropic assemblage of bodies. By this is meant that the statistical average of any nth order correlation function depends only on the difference coordinates of the points involved and is independent of the absolute orientation of this group of points. In such a case the tensor $\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)$ is an isotropic tensor function of $r = \mathbf{x}-\mathbf{x}_1$. The general form of this tensor is given in Batchelor [17] and we may immediately write

$$\mathscr{C}_{ijkl}(\mathbf{r}) = C_1r_ir_jr_kr_l+C_2r_ir_j\delta_{kl}+C_3\delta_{ij}r_kr_l+C_4(r_jr_k\delta_{il}+r_ir_k\delta_{jl}+r_lr_i\delta_{jk}+r_jr_l\delta_{ik})$$
$$+C_5\delta_{ij}\delta_{kl}+C_6(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}). \quad (38)$$

In writing this expression use has been made of the fact that $\mathscr{C}_{ijkl}$ is symmetric with respect to the first pair and the last pair of indices. The coefficients C_i are functions of r.

Theoretically one could calculate the coefficients appearing in equation (38) from the expressions developed in Section 2, once the assumption of isotropic statistics has been introduced. Alternately, one might view these coefficients as system parameters which are to be determined experimentally. We now consider the effect of statistical homogeneity and isotropy on the form of the governing equations.

Referring to equation (32) we note that the most important consequence on the difference coordinate is that the integrals appearing in this integro-differential equation are of the Faltung type. Thus, it is possible to introduce an integral transform such that the equation becomes algebraic in transform space. Introducing the Fourier transform and making use of the appropriate convolution integral the equation becomes, in **k** space,

$$ik_j\hat{\mathscr{C}}_{ijkl}\{\hat{\varepsilon}_{kl}\} = \hat{F}_i \qquad i = 1,2,3. \tag{39}$$

In the above, $\hat{}$ indicates the Fourier transform of the indicated quantity. In addition, we have

$$\{\hat{\varepsilon}_{kl}\} = -\frac{i}{2}(k_k\{\hat{u}_l\}+k_l\{\hat{u}_k\}). \tag{40}$$

Substituting equation (40) into equation (39) results in the following system of linear algebraic equations on the transform of displacement field.

$$(\hat{\mathscr{C}}_{irsj}k_rk_s)\{\hat{u}_j\} = \hat{F}_i. \tag{41}$$

The tensor $\hat{\mathscr{C}}_{irsj}$ is an isotropic tensor in **k** space and therefore has the same form as equation (38) with **k** replacing **r**. If we denote the coefficients in $\hat{\mathscr{C}}_{irsj}$ by $\hat{C}_i$; $i = 1, \ldots, 6$, then we may write

$$(\hat{\mathscr{C}}_{irsj}k_rk_s) = \hat{f}_1k_ik_j+\hat{f}_2k^2\delta_{ij}, \tag{42}$$

and

$$\hat{f}_1(k) = \hat{C}_1k^4+(\hat{C}_2+\hat{C}_3+3\hat{C}_4)k^2+\hat{C}_5+\hat{C}_6,$$
$$\hat{f}_2(k) = \hat{C}_4k^2+\hat{C}_6.$$

Use of equations (42) in (41) results in

$$(\hat{f}_1k_ik_j+\hat{f}_2k^2\delta_{ij})\{\hat{u}_j\} = \hat{F}_i. \tag{43}$$

The left hand side of equation (43a) is recognizable as the transform of the Navier operator of a homogeneous linearly elastic solid if one sets $\hat{f}_1 = \lambda+\mu$ and $\hat{f}_2 = \mu$. In transform space, then, the problem of determining the variations in the mean fields of a statistically homogeneous and isotropic sample of isotropic linearly elastic solids is identical in form to that of determining the variations in the fields of a single homogeneous and isotropic elastic solid. The difference between the two problems is that $\hat{f}_1$ and $\hat{f}_2$ are constants in the latter case, whereas they are functions of the magnitude of the transform variable **k** in the former.

In the small perturbation limit; i.e. $\{\lambda'^n\}/\{\lambda\}^n \ll 1$ and $\{\mu'^n\}/\{\mu\}^n \ll 1$; for all n; it is possible to obtain an explicit expression for $\hat{f}_1(k)$ and $\hat{f}_2(k)$. The calculations are straightforward but rather tedious and only an outline of the procedure will be given. $\mathscr{C}_{ijkl}(\mathbf{r})$ is given by equation (33) where D_{ijkl} and $E_{ijkl}(\mathbf{r})$ are given by equations (30) and (31). We define $\hat{\mathscr{C}}_{ijkl}(\mathbf{k})$ by

$$\hat{\mathscr{C}}_{ijkl}(\mathbf{k}) = \int \mathscr{C}_{ijkl}(\mathbf{r}) \exp(i\mathbf{k}\,.\,\mathbf{r})\,d\mathbf{r}, \tag{44}$$

insert $\mathscr{C}_{ijkl}(\mathbf{r})$, introduce spherical coordinates and integrate over angular coordinates.

The result gives

$$\hat{C}_1 = \frac{30\{\lambda+\mu\}}{\{\mu\}\{\lambda+2\mu\}}\frac{1}{k^4}\int_0^\infty \frac{C_{\mu\mu}(r)}{r}M_1(kr)\,dr,$$

$$\hat{C}_2 = \frac{6\{\lambda+\mu\}}{\{\mu\}\{\lambda+2\mu\}}\frac{1}{k^2}\int_0^\infty \frac{C_{\mu\mu}(r)}{r}M_2(kr)\,dr + \frac{6}{\{\lambda+2\mu\}}\frac{1}{k^2}\int_0^\infty \frac{C_{\mu\lambda}(r)}{r}M_3(kr)\,dr,$$

$$\hat{C}_3 = \frac{6\{\lambda+\mu\}}{\{\mu\}\{\lambda+2\mu\}}\frac{1}{k^2}\int_0^\infty \frac{C_{\mu\mu}(r)}{r}M_2(kr)\,dr + \frac{6}{\{\lambda+2\mu\}}\frac{1}{k^2}\int_0^\infty \frac{C_{\lambda\mu}(r)}{r}M_3(kr)\,dr$$

$$\hat{C}_4 = \frac{3}{\{\lambda+2\mu\}}\frac{1}{k^2}\int_0^\infty \frac{C_{\mu\mu}(r)}{r}\left[M_3(kr)+\frac{\{\lambda+\mu\}}{\{\mu\}}M_4(kr)\right]dr,$$

$$\hat{C}_5 = \{\lambda\} - \left[\{\lambda'^2\} + \frac{4}{3}\{\lambda'\mu'\} - \frac{4\{\lambda+\mu\}}{15\{\mu\}}\{\mu'^2\}\right]\Big/\{\lambda+2\mu\}$$
$$-\frac{2}{\{\lambda+2\mu\}}\int_0^\infty\left[\frac{C_{\lambda\mu}(r)+C_{\mu\lambda}(r)}{r}\right]M_3(kr)\,dr + \frac{\{\lambda+\mu\}}{\{\mu\}\{\lambda+2\mu\}}\int_0^\infty \frac{C_{\mu\mu}(r)}{r}M_5(kr)\,dr,$$

$$\hat{C}_6 = \{\mu\} - \frac{2\{3\lambda+8\mu\}}{15\{\mu\}\{\lambda+2\mu\}}\{\mu'^2\} + \frac{1}{\{\lambda+2\mu\}}\int_0^\infty \frac{C_{\mu\mu}(r)}{r}\left[\frac{3\{\lambda+\mu\}}{\{\mu\}}M_6(kr) - 2M_3(kr)\right]dr, \tag{45}$$

where

$$M_1(y) = \left(\frac{1}{y}-\frac{45}{y^3}+\frac{105}{y^5}\right)\sin y + \left(\frac{10}{y^2}-\frac{105}{y^4}\right)\cos y,$$

$$M_2(y) = \left(\frac{-1}{y}+\frac{33}{y^3}-\frac{75}{y^5}\right)\sin y - \left(\frac{8}{y^2}-\frac{75}{y^4}\right)\cos y,$$

$$M_3(y) = \left(\frac{1}{y}-\frac{3}{y^3}\right)\sin y + \frac{3}{y^2}\cos y,$$

$$M_4(y) = \left(\frac{-1}{y}+\frac{63}{y^3}-\frac{150}{y^5}\right)\sin y - \left(\frac{13}{y^2}-\frac{150}{y^4}\right)\cos y,$$

$$M_5(y) = \left(\frac{2}{y}-\frac{42}{y^3}+\frac{90}{y^5}\right)\sin y + \left(\frac{12}{y^2}-\frac{90}{y^4}\right)\cos y,$$

and

$$M_6(y) = \left(\frac{-12}{y^3}+\frac{30}{y^5}\right)\sin y + \left(\frac{2}{y^2}-\frac{30}{y^4}\right)\cos y. \tag{46}$$

Using the above expressions, we may easily consider the limiting case as $k \to 0$. The result obtained is

$$\hat{\mathscr{C}}_{ijkl} \to \left(\{\lambda\} - \frac{\{\lambda'^2\}+\frac{4}{3}\{\lambda'\mu'\}-\frac{4\{\lambda+\mu\}}{15\{\mu\}}\{\mu'^2\}}{\{\lambda+2\mu\}}\right)\delta_{ij}\delta_{kl}$$
$$+\left(\{\mu\} - \frac{2\{3\lambda+8\mu\}}{15\{\mu\}\{\lambda+2\mu\}}\{\mu'^2\}\right)(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}). \tag{47}$$

This limit is of interest since it corresponds to the limit in which the mean stress and mean strain are homogeneous. Thus, it is natural to relate the limit of $\hat{\mathscr{C}}_{ijkl}(\mathbf{k})$ as $\mathbf{k}$ approaches zero to "effective" properties. We may thus write

$$\hat{\mathscr{C}}_{ijkl}(\mathbf{k}) \rightarrow \lambda^*\delta_{ij}\delta_{kl}+\mu^*(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}), \tag{48}$$

where the small perturbation limits of λ^* and μ^* are given by equation (47). These perturbation limits for λ^* and μ^* are in agreement with those reported by McCoy [2], and with k^* (where $k = \lambda+\frac{2}{3}\mu$, the bulk modulus) that was reported by Beran and Molyneux [1].

We consider the other limit (i.e. $k \rightarrow \infty$). For k large enough, $M_i(kr)$ falls off sufficiently fast relative to the $C(r)$'s so that the latter may be taken as given by their values at $r = 0$. Then, the integration over r may be carried out and we find

$$\begin{aligned}\hat{\mathscr{C}}_{ijkl} \rightarrow & \frac{-\{\lambda+\mu\}}{\{\mu\}\{\lambda+2\mu\}}\{\mu'^2\}k_ik_jk_kk_l/k^4-\frac{2\{\lambda'\mu'\}}{\{\lambda+2\mu\}}(k_ik_j\delta_{kl}+\delta_{ij}k_kk_l)/k^2 \\ & -\frac{\{\mu'^2\}}{\{\mu\}}(\delta_{ik}k_jk_l+\delta_{il}k_jk_k+\delta_{jk}k_ik_l+\delta_{jl}k_ik_k)/k^2 \\ & +\left[\{\lambda\}-\frac{\{\lambda'^2\}}{\{\lambda+2\mu\}}\right]\delta_{ij}\delta_{kl}+\{\mu\}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}).\end{aligned} \tag{49}$$

Next substituting equations (45) into equations (43) gives

$$\begin{aligned}\hat{f}_1(k) = & \{\lambda+\mu\}-\left[\{\lambda'^2\}+\frac{4}{3}\{\lambda'\mu'\}+\frac{2\{\lambda+6\mu\}}{15\{\mu\}}\{\mu'^2\}\right]\Big/\{\lambda+2\mu\} \\ & +\frac{1}{\{\lambda+2\mu\}}\int_0^\infty\frac{C_{\mu\mu}(r)}{r}\left[7M_3(kr)+\frac{\{\lambda+\mu\}}{\{\mu\}}M_7(kr)\right]\mathrm{d}r \\ & +\frac{4}{\{\lambda+2\mu\}}\int_0^\infty\frac{C_{\mu\lambda}(r)+C_{\lambda\mu}(r)}{r}M_3(kr)\,\mathrm{d}r,\end{aligned}$$

$$\hat{f}_2(k) = \{\mu\}-\frac{2\{3\lambda+8\mu\}}{15\{\mu\}\{\lambda+2\mu\}}\{\mu'^2\}+\frac{1}{\{\lambda+2\mu\}}\int_0^\infty\frac{C_{\mu\mu}(r)}{r}\left[M_3(kr)+\frac{3\{\lambda+\mu\}}{\{\mu\}}M_8(kr)\right]\mathrm{d}r, \tag{50}$$

where

$$M_7(y) = \left(\frac{11}{y}-\frac{465}{y^3}+\frac{1080}{y^5}\right)\sin y+\left(\frac{105}{y^2}-\frac{1080}{y^4}\right)\cos y,$$

and

$$M_8(y) = \left(\frac{-1}{y}+\frac{51}{y^3}-\frac{120}{y^5}\right)\sin y+\left(\frac{-11}{y^2}+\frac{120}{y^4}\right)\cos y. \tag{51}$$

The same limits of k approaching zero and k approaching infinity may be taken. The results are:

$$\hat{f}_1(k\rightarrow 0) \rightarrow \{\lambda+\mu\}-\left[\frac{\{\lambda'^2\}+\frac{4}{3}\{\lambda'\mu'\}+\frac{2\{\lambda+6\mu\}}{15\{\mu\}}\{\mu'^2\}}{\{\lambda+2\mu\}}\right],$$

$$\hat{f}_2(k\rightarrow 0) \rightarrow \{\mu\}-\left[\frac{2\{3\lambda+8\mu\}\{\mu'^2\}}{15\{\mu\}\{\lambda+2\mu\}}\right], \tag{52}$$

and

$$\hat{f}_1(k \to \infty) \to \{\lambda+\mu\} - \left[\frac{\{\lambda'^2\}+4\{\lambda'\mu'\} - \dfrac{\{\lambda-2\mu\}}{\{\mu\}}\{\mu'^2\}}{\{\lambda+2\mu\}}\right]$$

$$\hat{f}_2(k \to \infty) \to \{\mu\} - \frac{\{\mu'^2\}}{\{\mu\}}. \tag{53}$$

The limits as k approaches zero are easily recognized to be the small perturbation expressions for $(\lambda^* + \mu^*)$ and μ^*, respectively. This, of course, is to be expected. The limits as k increases without bound may be identified, in the limit of small perturbations, with the results of the following sequence of operations. Let λ and μ be independent of $\mathbf{r}$ but vary from sample to sample. Invert the tensor

$$[(\lambda+\mu)k_i k_j + \mu k^2 \delta_{ij}],$$

take the statistical average of the result, and finally invert the result.

This latter result might also be expected if one uses the following reasoning. Inverting the set of equations represented by equation (39) gives

$$\{\hat{u}_j\} = [\hat{\mathscr{C}}_{irsj} k_r k_s]^{-1} \hat{F}_i. \tag{54}$$

The limit $k \to \infty$ corresponds to the limit $r \to 0$ in the transform domain. When $r \to 0$ the solution may be obtained by considering a sequence of different samples all of which have elastic moduli that do not vary with position but only vary from sample to sample. The constants in this case are obtained by the procedure in the above paragraph.

4. SLOWLY VARYING MEAN FIELDS

We should now like to consider the simplifications that can be introduced when the smallest characteristic length associated with the variations in the mean field, i.e. L, are much greater than the largest characteristic length associated with the random variations in the material properties, i.e. l.

As shown in Section 2, if one expands $\{\varepsilon_{kl}(\mathbf{x}_1)\}$ in a power series about the point $\mathbf{x}_1 = \mathbf{x}$, one obtains, in place of the integral function relating the average stress at a point to the mean strain at every point, the series expression

$$\{\tau_{ij}\} = C^*_{ijkl}\{\varepsilon_{kl}\} + \mathscr{D}_{ijklm}[\partial_n\{\varepsilon_{kl}\}] + \mathscr{E}_{ijklmn}[\partial_m\partial_n\{\varepsilon_{kl}(\mathbf{x})\}] + \dots. \tag{55}$$

The formal expression defining $\mathscr{D}_{ijklm}$, $\mathscr{E}_{ijklmn}$ etc. show that each of these terms may be viewed as tensor products of C^*_{ijkl} and weighted position vectors from the point $\mathbf{x}$ to the point $\mathbf{x}_1$. Since the weighting function is $\mathscr{C}_{ijkl}(\mathbf{x}, \mathbf{x}_1)$, the magnitude of these vectors are of order l. If the smallest characteristic length associated with the fluctuations in $\{\varepsilon_{kl}\}$ is L, the nth term in this series is of order $(l/L)^n$. Thus, if $l/L \ll 1$, it is possible to define various order approximations by truncating this series after differing numbers of terms.

Assuming statistically homogeneous and isotropic statistics, the various tensors in equation (55) must be constant isotropic tensors. Again referring to Batchelor [17], we may immediately express this requirement by

$$C^*_{ijkl} = \lambda^*\delta_{ij}\delta_{kl} + \mu^*(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{56a}$$

$$\mathscr{D}_{ijklm} = 0 \tag{56b}$$

$$\begin{aligned}\mathscr{E}_{ijklmn} = {} & a_1\delta_{ij}\delta_{kl}\delta_{mn}+a_2\delta_{ij}(\delta_{km}\delta_{ln}+\delta_{kn}\delta_{lm})+a_3\delta_{mn}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+a_4\delta_{kl}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})\\ & +a_5(\delta_{ik}\delta_{ln}\delta_{jm}+\delta_{il}\delta_{kn}\delta_{jm}+\delta_{ik}\delta_{lm}\delta_{jn}+\delta_{il}\delta_{km}\delta_{jn}\\ & +\delta_{jk}\delta_{ln}\delta_{im}+\delta_{jl}\delta_{kn}\delta_{im}+\delta_{jk}\delta_{lm}\delta_{in}+\delta_{jl}\delta_{km}\delta_{in}). \end{aligned} \tag{56c}$$

(Note $\mathscr{E}_{ijklmn}$ is symmetric in i and j, in k and l and in m and n.) Substitution of equations (56) in equation (57) gives

$$\begin{aligned}\{\tau_{ij}\} = {} & \lambda^*\{\varepsilon_{kk}\}\delta_{ij}+2\mu^*\{\varepsilon_{ij}\}+\partial_m\partial_m[a_1\{\varepsilon_{kk}\}\delta_{ij}+2a_3\{\varepsilon_{ij}\}]+2a_2\partial_k\partial_l\{\varepsilon_{kl}\}\delta_{ij}\\ & +2a_4\partial_i\partial_j\{\varepsilon_{kk}\}+4a_5[\partial_j\partial_k\{\varepsilon_{ki}\}+\partial_i\partial_k\{\varepsilon_{kj}\}]+\dots. \end{aligned} \tag{57}$$

Truncating equation (57) after the first set of terms, making use of equation (34c) relating the mean strain field to the mean displacement field, and substituting the result into equation (34a) gives, as the equation governing $\{u_i\}$, the homogeneous Navier equation with λ and μ replaced by λ^* and μ^*.

Truncating equation (57) after the first two sets of terms, making use of equation (34c), and substituting the result into equation (34a) gives the following equations on $\{u_i\}$.

$$(\lambda^*+2\mu^*)(1-l_1^2\nabla^2)\nabla\,\nabla\,.\,\{\mathbf{u}\}-\mu^*(1-l_2^2\nabla^2)\nabla\times\nabla\times\{\mathbf{u}\}+\mathbf{F}=0 \tag{58}$$

where

$$l_1^2 = -(a_1+2a_2+2a_3+2a_4+8a_5)/(\lambda^*+2\mu^*),$$

and

$$l_2^2 = -(a_3+2a_5)/\mu^*. \tag{59}$$

Equation (58) may be recognized as a linearized version of the first strain gradient theory of Troupin [15] as it is presented by Mindlin and Eshel [18]. The latter grouped the new material parameters as indicated by equation (59) since it can be shown that a necessary condition for the positive definiteness of the strain energy density defined in the first gradient theory is the positiveness of l_1^2 and l_2^2.

Using the expressions developed in Sections 2 and 3, it is possible, theoretically at least, to calculate the quantities l_1^2 and l_2^2. For a statistical sample in which the media are only weakly inhomogeneous such a calculation is even practical. Thus, we present the first term correction for a weakly inhomogeneous media. No details of the tedious but straightforward calculations are given:

$$\begin{aligned} a_1 &= -\frac{2(\sigma_{\lambda\mu}+\sigma_{\mu\lambda})}{15\{\lambda+2\mu\}}+\frac{8\{\lambda+\mu\}}{105\{\mu\}\{\lambda+2\mu\}}\sigma_{\mu\mu},\\ a_2 &= +\frac{\sigma_{\lambda\mu}}{5\{\lambda+2\mu\}}-\frac{2\{\lambda+\mu\}}{35\{\mu\}\{\lambda+2\mu\}}\sigma_{\mu\mu},\\ a_3 &= -\frac{2\{3\lambda+10\mu\}}{105\{\mu\}\{\lambda+2\mu\}}\sigma_{\mu\mu},\\ a_4 &= +\frac{\sigma_{\mu\lambda}}{5\{\lambda+2\mu\}}-\frac{2\{\lambda+\mu\}}{35\{\mu\}\{\lambda+2\mu\}}\sigma_{\mu\mu},\\ a_5 &= +\frac{\{3\lambda+10\mu\}}{70\{\mu\}\{\lambda+2\mu\}}\sigma_{\mu\mu}, \end{aligned} \tag{60}$$

where

$$\sigma_{\lambda\mu} = \int_0^\infty rC_{\lambda\mu}(r)\,dr \quad \text{with similar expressions for } \sigma_{\mu\lambda} \text{ and } \sigma_{\mu\mu}.$$

Therefore

$$l_1^2 = -\left[\frac{4}{15}(\sigma_{\lambda\mu}+\sigma_{\mu\lambda})+\frac{8\{\lambda+8\mu\}}{105\{\mu\}}\sigma_{\mu\mu}\right]\Big/\{\lambda+2\mu\}(\lambda^*+2\mu^*),$$

and

$$l_2^2 = -\frac{1}{105}\frac{\{3\lambda+10\mu\}}{\{\mu\}}\sigma_{\mu\mu}\Big/\{\lambda+2\mu\}\mu^*. \tag{61}$$

For homogeneous and isotropic statistics the function $\sigma_{\mu\mu}$ is positive and thus, we see that in the small perturbation limit the parameter l_2^2 is a negative number. This is in disagreement with the positive definiteness of the strain energy density defined in the first strain gradient theory. From this one can draw the conclusion that the strain gradient theory is not a valid mathematical model for the physical problem being treated here. A more detailed treatment of this point will be presented in the subsequent note. We note that if there are no fluctuations in μ that $l_1 = l = 0$.

Finally it might be well at this point to emphasize the distinction between the interpretation to be given to equations (58) from that given the similar equations when they are obtained from first gradient theory. $\{\mu\}$ in equation (58) represents the statistical average of the displacement fields which exist in a statistical sample of heterogeneous linear elastic solids. In the first gradient theory the analogous quantity represents the displacement field in a single homogeneous solid when gradients of the strain tensor are allowed to enter the internal energy density function in addition to the strain tensor itself. It is only in the situation in which an ergodic hypothesis can be invoked equating statistical averages to local volume averages that one can extract any information from $\{\mathbf{u}\}$ regarding the response of a single solid. In the next section the validity of the ergodic hypothesis is discussed and it is argued that the validity requires $l/L \ll 1$, which is the same required for equation (58) to be a valid approximation of equations (34c).

It seems clear that the retention of higher order terms in equation (57) would result in equations analogous to the higher order gradient theories. (See Cauchy [19], Mindlin [20], and Green and Rivlin [21].

5. DISCUSSION OF ENSEMBLE AND VOLUME AVERAGING

The following discussion on the relationship between ensemble and volume averaging exactly parallels our discussion of this problem on dielectrics given in Ref. [14].

The meaning of $\{\varepsilon_{ij}(\mathbf{x})\}$ as defined in this paper is unambiguous. We consider a family of linearly elastic solids for which there are associated probability distributions for the fields $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$. Each member of the family is, in turn, subjected to an identical excitation $F(\mathbf{x})$ and the value of the elastic strain tensor is either measured or calculated at the identical point denoted by $\mathbf{x}$. $\{\varepsilon_{ij}(\mathbf{x})\}$ is the result of averaging all of the values so measured (or calculated).

It is not so clear how to infer any information from $\{\varepsilon_{ij}(\mathbf{x})\}$ that is useful for discussing the response of a single linearly elastic solid, the material properties of which vary with position in space in a manner about which we have only limited information. If the random vector field $\{\varepsilon_{ij}(\mathbf{x})\}$ is defined over an infinite region of space and if it is statistically homogeneous, i.e. $\{\varepsilon_{ij}(\mathbf{x})\}$ is not a function of position, then it is possible to invoke an ergodic hypothesis. The ergodic hypothesis equates the ensemble average with a spatial volume average. Thus, one may view $\{\varepsilon_{ij}(\mathbf{x})\}$ as a spatial average of the elastic strain tensor that exists in a single solid.

In the case in which the random tensor field is not statistically homogeneous the conditions justify the invoking of an ergodic hypothesis are not present. Still, one could argue that if $\{\varepsilon_{ij}(\mathbf{x})\}$ varied "slowly" with position in space then the conditions necessary for an ergodic hypothesis to be valid are approximately present. In such a case it could be hoped that some information of the response of the single medium problem might be inferred from $\{\varepsilon_{ij}(\mathbf{x})\}$.

To give some precision to $\{\varepsilon_{ij}(\mathbf{x})\}$ varying "slowly" with position and what one might infer from $\{\varepsilon_{ij}(\mathbf{x})\}$, one might imagine that the variations in $\varepsilon_{ij}(\mathbf{x})$ are seen over two scales. On one scale one could discern details of the variations of the material properties. On this scale (the inner scale) the overall dimensions of the solid and any characteristic length associated with the forcing of the solid appear to be infinitely large. On the second scale (the outer scale) one can make measurements of the overall dimensions of the body and of characteristic lengths associated with the forcing. One this scale the fluctuations in the material properties with position in space are too rapid to be discernible. The variations of $\varepsilon_{ij}(\mathbf{x})$ with distance measured on the inner scale are variations due to the variations in the material properties. The variations of $\varepsilon_{ij}(\mathbf{x})$ with distance measured on the outer scale arise due to the finiteness of the solid and/or the finiteness of all characteristic lengths associated with the forcing. If $\{\varepsilon_{ij}(\mathbf{x})\}$ does not vary appreciably with a change in position of any length measured on the inner scale, then the conditions for the justification of an ergodic hypothesis are present on this scale. Hence, $\{\varepsilon_{ij}(\mathbf{x})\}$ may be associated with a local spatial volume average over a region with dimensions very large compared to the inner scale.

In a specific problem, the length defining the inner scale, which we may denote by l_i, will be given by some correlation length associated with the variations in the material properties. For example, the correlation length associated with $C_{\mu\mu}(r)$, $C_{\mu\lambda}(r)$ and $C_{\lambda\lambda}(r)$. The length defining the outer scale, which we may denote by L_0, has already been defined as the smallest characteristic length that can either be associated with the overall geometry or with the forcing mechanism (i.e. L_F). If $l_i/L_0 \ll 1$, one can equate $\{\varepsilon_{ij}(\mathbf{x})\}$ to a local volume average taken over a region which is large compared to l_i but small compared to L_0.

For problems in which two clearly discernible length scales are not present it is not possible to extract any deterministic information regarding the response of a single medium from statistical averages such as $\{\varepsilon_{ij}(\mathbf{x})\}$. In general equation (34) only makes sense if viewed from an ensemble point of view. If, however, $l_i/L_F \ll 1$ (e.g. $l/L_F \ll 1$) the problem may be viewed from either an ensemble averaged or volume averaged point of view. Equation (58) admits of either interpretation.

6. EFFECT OF THE PRESENCE OF A BOUNDARY

We should now like to turn to the effect of the presence of a boundary on the equation obtained for $\{\varepsilon_{kl}(\mathbf{x})\}$. In reviewing the development of this equation one may easily see that such an effect appears in the form of the Green's function K_{klm}. For the infinite solid, the appropriate Green's function is given by the Kelvin solution for a force applied at a point. For a bounded solid this function must be suitably modified to insure that the desired conditions are met on the surface defining the limits of the solid. Unfortunately, the appropriate modifications that must be introduced are known only for the simplest geometries and even then it is usually necessary to require that the prescribed conditions be of a certain type. These restrictions, however, do not prevent our gaining some insight into the effect of the presence of boundaries on the "field" equation governing $\{\varepsilon_{kl}(\mathbf{x})\}$.

Consider the following problem. A force is applied at a point in a semi-infinite solid that is directed away from the plane surface that serves as a boundary. The conditions to be satisfied on this boundary are that the component of the displacement vector normal to the surface and the components of the traction vector tangential to the surface vanish. The solution to this problem is readily obtained by superposing two Kelvin solutions; one for the described force acting at the described point and one for the negative of the described force acting at the image of the described point (the image is taken about the plane boundary). Thus, the presence of this special boundary to this special type forcing is contained in the contribution that arises due to the presence of the image point and the fact that the region of integration is different. We should now like to investigate the size of the contribution of the image point to the equation governing $\{\varepsilon_{kl}\}$ relative to the contribution of the primary source point. We also consider the effect of integration over the half space instead of the full space as was done in the infinite medium problem.

The contribution to the Green's function of the primary source point is given by equation (18) with r_i representing the position vector of the field point relative to the primary source point. For the specific problem under consideration, it is to be understood that this Green's function has been specialized to account for the force being directed normal to the boundary plane. Similarly, the contribution to the Green's function of the image source point is also given by equation (18) (with an appropriate sign change) provided one takes r_i, in this case, to represent the position vector of the field point relative to the image source point.

The Green's function enters the governing equation on $\{\varepsilon_{kl}(\mathbf{x})\}$ through the expression $\{C'_{ijkl}(\mathbf{x})\varepsilon'_{kl}(\mathbf{x})\}$. The manner in which this expression depends on the Green's function is given by equations (11)–(13). For the half space problem being treated, the expressions are the same as those given there if one replaces

$$K_{klr}(\mathbf{x}, \mathbf{x}_1) = K_{klr}(\mathbf{x}, \mathbf{x}_1; \mathbf{x}_1) + K_{klr}(\mathbf{x}, \mathbf{x}_1; \mathbf{x}_2), \tag{62}$$

where $K_{klr}(\mathbf{x}, \mathbf{x}_1; \mathbf{x}_1)$ is the contribution from the primary source point and $K_{klr}(\mathbf{x}, \mathbf{x}_1; \mathbf{x}_2)$ is the contribution from the image source point. In addition, of course, for the semi-infinite body the region of integration is the half space occupied by the body. Turning first to equation (12), we note that the integrand contains the two point correlation function $\{C'_{ijkl}(\mathbf{x})C'_{rspq}(\mathbf{x}_1)\}$. For such a function it is possible to introduce a characteristic length, say l, and state that the function falls off rapidly for $|x - x_1| > l$. The fact allows us to introduce the following simplifications into the expression for $\{C'_{ijkl}(\mathbf{x})\varepsilon'^{(1)}_{kl}(\mathbf{x})\}$. First of all, if the field point $\mathbf{x}$ is located sufficiently far from the boundary (relative to length l), the contribution from the image source point is negligible when compared to the contribution from the primary source point. To draw this conclusion one needs to consider the relative sizes of $K_{klr}(\mathbf{x}, \mathbf{x}_1; \mathbf{x}_1)$ and $K_{klr}(\mathbf{x}, \mathbf{x}_1; \mathbf{x}_2)$ in the region of $\mathbf{x}_1$ space in which $\{C'_{ijkl}(\mathbf{x})C'_{rspq}(\mathbf{x}_1)\}$ is not negligible, namely a small region about the field point $\mathbf{x}$. From the form of equation (18), it may be concluded that the size of $K_{klr}(\mathbf{x}, \mathbf{x}_1; \mathbf{x}_2)$ relative to the size of $K_{klr}(\mathbf{x}, \mathbf{x}_1; \mathbf{x}_1)$

where $\mathbf{x}_1$ is to be understood to locate a point in this region, is of order $(l/L)^2$ where L, in this instance, denotes the distance of the source point $\mathbf{x}$ from the boundary. Thus, except for a certain region of the boundary, one may neglect the contribution from the image source point. Secondly, subject to the same restriction that the field point located sufficiently far from the boundary, the domain of integration for the contribution from the primary source point may be extended to cover the entire space. This is valid since the integrand is negligible throughout the added half space. Thus, for the half space problem being treated, equation (12) as calculated assuming an infinite medium represents a valid approximation. Turning to equation (13), we can draw the same conclusions by means of the same arguments if we assume that $\{C'_{ijkl}(\mathbf{x})C_{rspq}(\mathbf{x}_1)\varepsilon'^{(n-1)}_{pq}(\mathbf{x}_1)\}$ decays in a manner similar to $\{C'_{ijkl}(\mathbf{x})C'_{rspq}(\mathbf{x}_1)\}$. That is, if we assume that a correlation length may be assigned to $\{C'_{ijkl}(\mathbf{x})C'_{rspq}(\mathbf{x}_1)\varepsilon'^{(n-1)}_{pq}(\mathbf{x}_1)\}$ such that its value falls off rapidly for $|x-x_1|$ greater than this length. Physically it appears that such an assumption is valid since it is difficult to imagine that $\varepsilon'_{pq}(\mathbf{x}_1)$ is strongly correlated to $C'_{ijkl}(\mathbf{x})$ when the distance between $\mathbf{x}_1$ and $\mathbf{x}$ becomes large.

Based on the above reasoning, therefore, it seems reasonable to conclude that the presence of the boundary plane of the special type investigated for the special loading investigated offers a significant contribution to the equation on $\{\varepsilon_{kl}(\mathbf{x})\}$ only within a layer of the boundary. The extension to other loadings and to other type conditions on boundaries with different geometries would proceed with similar arguments. The first step would be to construct the appropriate Green's function for the boundary being introduced and then continue as above. The construction of the Green's function may sometimes be accomplished by a process of synthesis in which differing combinations of Kelvin's solution are superposed. For example, for the point force in a semi-infinite solid with a traction free surface, Mindlin [22] accomplished this synthesis with the following results. For the force directed normal to the boundary the solution may be given by the six nuclei of strain which, in an infinite solid, represent: (1) a single force at the primary source point; (2) a single force at the image source point; (3) a double force (i.e. two equal and opposite forces) directed normal to the boundary also at the image source point; (4) a center of compression (i.e. three double forces directed along the three axes) at the image source point; (5) a doublet (i.e. double center of compression and dilatation) at the image source point; and (6) a line of centers of compression running from the image source point out to infinity along the line normal to the boundary. With this result it is not difficult to proceed as in the case of the mixed boundary value problem (specification of one component of the displacement vector and two of the traction vector) and draw the same conclusions. Mindlin also presented the same synthesis for the force directed parallel to the boundary. These results can be used to draw the same conclusion for that problem, namely that the presence of the boundary is significant only within a layer of the boundary. Finally, the extension to other than plane boundaries is difficult because of the difficulty in constructing the Green's function. It seems reasonable to assume, however, that so long as the curvature at every point of the boundary is not too great the conclusions drawn here will still be valid.

REFERENCES

[1] M. Beran and J. Molyneux, *Q. appl. Math.* **24**, 107 (1966).
[2] J. J. McCoy, *Proc. 6th Annual Meeting Soc. Engng. Sci.* (1966).
[3] E. Kroner, *J. Mech. Phys. Solids* **15**, 319 (1967).

[4] Z. HASHIN, *Appl. Mech. Rev.* **17**, 1 (1964).
[5] M. BERAN, *Statistical Continuum Theories*. Wiley (1968).
[6] V. A. LOMAKIN, *J. appl. Math. Mech.* **29**, 889 (1965).
[7] V. A. LOMAKIN, *Engng J. Mech. Solids* **3**, 72 (1966).
[8] V. A. LOMAKIN, *Proc. 2nd Conf. on Problems of the Validity of Theories of Structural Mechanics*. Vilnius (1968).
[9] V. A. LOMAKIN, to be published (1970).
[10] J. B. KELLER, *Proc. Symp. appl. Math.* **16**, 145 (1964).
[11] F. KARAL and J. B. KELLER, *J. Math. Phys.* **5**, 537 (1964).
[12] L. KNOPOFF and J. A. HUDSON, *J. acoust. Soc. Am.* **42**, 1, 18 (1969).
[13] J. A. HUDSON, *Q. J. Mech. appl. Math.* **21**, 487 (1968).
[14] M. J. BERAN and J. J. McCOY, *Q. appl. Math.* (1970). In press.
[15] R. A. TOUPIN, *Arch. Rat. Mech. Analysis* **11**, 385 (1962).
[16] A. E. H. LOVE, *The Mathematical Theory of Elasticity*. Dover (1944).
[17] G. K. BATCHELOR, *The Theory of Homogeneous Turbulence*. Cambridge University Press (1953).
[18] R. D. MINDLIN and N. N. ESHEL, *Int. J. Solids Struct.* **4**, 109 (1968).
[19] A. L. CANCHY, *C. r. hebd. Seanc. Acad. Sci., Paris* **32**, 323 (1851).
[20] R. D. MINDLIN, *Int. J. Solids Struct.* **1**, 417 (1965).
[21] A. E. GREEN and R. S. RIVLIN, *Arch. rat. Mech. Analysis* **16**, 325 (1964).
[22] R. D. MINDLIN, *Physics* **7**, 195 (1936).

(*Received 18th November* 1969)

Абстракт—Предметом работы является определение значения /то есть статистического среднего/ величин поля в статистическом образце неоднородных линейно упругих твердых тел. Полученная формула определяет $\{u_i(x)\}$, где u_i обозначает поле перемещений, а скобки обозначают среднее знцле нце совокупности. Такая формулировка представлена выражениями бесконечной последовательности функций корреляции, связанных со слчайными свойствами материала. Тогда как, эта формулировка справедлива только для случая, касающегося бесконечных твердых тел, пример используется для указания, что эффект границы имеет значение при формуицревке только совместно с пограничным слоем. Толщина слоя определяется найбольшой длиной корреляции связанной с полем, описывающим свойства материала.

Формулировка обсуждается, подробно, для слчая статистически однородного и статистически изотропного образца локально изотропных твердых тел. Даются формула в конечном виде для предельного случая слабо неоднородных тел. Исследуется также случай медленного изменяющегося поля, то есть изменения выражения $\{u_i(x)x\}$, которые медленно зависят от изменений свойств материала. Указывается, что в данном случае, использование эффективных параметров материала дает результат для действительного приближения первого пирчуки. Корректура первого рода для этихже самых ограничений является теорией градиента деформации первого пирчуки.